

Equality of Distance Packing Numbers

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Abstract

We characterize the graphs for which the independence number equals the packing number. As a consequence we obtain simple structural descriptions of the graphs for which (i) the distance- k -packing number equals the distance- $2k$ -packing number, and (ii) the distance- k -matching number equals the distance- $2k$ -matching number. This last result considerably simplifies and extends previous results of Cameron and Walker (The graphs with maximum induced matching and maximum matching the same size, Discrete Math. 299 (2005) 49-55). For positive integers k_1 and k_2 with $k_1 < k_2$ and $\lceil (3k_2 + 1)/2 \rceil \leq 2k_1 + 1$, we prove that it is NP-hard to determine for a given graph whether its distance- k_1 -packing number equals its distance- k_2 -packing number.

Keywords: independent set; packing; matching; induced matching

1 Introduction

Induced matchings in graphs were introduced by Stockmeyer and Vazirani [12] as a variant of ordinary matchings. While the structure and algorithmic properties of ordinary matchings are well understood [11], induced matchings are algorithmically very hard [4, 7, 12]. Many efficient algorithms for finding maximum induced matchings exploit the fact that induced matchings correspond to independent sets of the square of the line graph [1–3, 5, 9]. In [10] Kobler and Rotics showed that the graphs where the matching number and the induced matching number coincide can be recognized efficiently. Their result was extended by Cameron and Walker [6] who gave a complete structural description of these graph. In [8] we generalized some results from [6, 10] to distance- k -matchings and simplified the original proofs. In the present paper we present much more general results systematically exploiting the above-mentioned relation between matchings and independent sets in line graphs. Our main result is a very simple characterization of the graphs for which the independence number equals the packing number. An immediate consequence of this result is a complete structural description of the graphs for which the distance- k -matching number equals the distance- $2k$ -matching number. It follows immediately that such graphs can be recognized by a very simple efficient algorithm. We establish further results relating distance packing numbers and discuss related open problems.

Before we proceed to the results, we recall some terminology. We consider finite, simple, and undirected graphs. Let G be a graph. A set P of vertices of G is a k -packing of G for some positive integer k if every two distinct vertices in P have distance more than k in G . The k -packing number $\rho_k(G)$ of G is the maximum cardinality of a k -packing of G , and a k -packing of cardinality $\rho_k(G)$ is *maximum*. Using this terminology, independent sets correspond to 1-packings and the independence number $\alpha(G)$ coincides with $\rho_1(G)$. We denote the *line graph* of G by $L(G)$ and the k -th *power* of G for some positive integer k by G^k . Since matchings of G correspond to independent sets of $L(G)$, the matching number $\nu(G)$ equals $\rho_1(L(G))$. Similarly, since induced matchings of G correspond to 2-packings of $L(G)$, the induced matching number $\nu_2(G)$ equals $\rho_2(L(G))$. More generally, a set M of edges of G is a k -matching of G if it is a k -packing of $L(G)$. The k -matching number $\nu_k(G)$ and *maximum k -matchings* are defined in the obvious way. Clearly, a set P is a k_1 -packing of G^{k_2} for some positive integers k_1 and k_2 if and only if it is a $k_1 k_2$ -packing of G , that is, $\rho_{k_1}(G^{k_2}) = \rho_1(G^{k_1 k_2})$. A vertex u of G is *simplicial* if $N_G[u]$ is complete. Two distinct vertices u and v of G are *twins* if $N_G[u] = N_G[v]$. Let $S(G)$ be the set of simplicial vertices of G . Let $\mathcal{S}(G)$ be the partition of $S(G)$ where two simplicial vertices belong to the same partite set if and only if they are twins. A *transversal* of $\mathcal{S}(G)$ is a set of simplicial vertices that contains exactly one vertex from each partite set of the partition $\mathcal{S}(G)$. Note that the subgraph of G induced by $S(G)$ is a union of cliques, and that $\mathcal{S}(G)$ is the collection of the vertex sets of these cliques. In particular, every transversal of $\mathcal{S}(G)$ is independent.

2 Results

We immediately proceed to the characterization of the graphs for which the independence number equals the packing number.

Theorem 1 *A graph G satisfies $\rho_1(G) = \rho_2(G)$ if and only if*

- (i) *a set of vertices of G is a maximum 2-packing if and only if it is a transversal of $\mathcal{S}(G)$, and*
- (ii) *for every transversal P of $\mathcal{S}(G)$, the sets $N_G[u]$ for u in P partition $V(G)$.*

Proof: Let G be a graph.

In order to prove the sufficiency, let G satisfy (i) and (ii). Let P be a transversal of $\mathcal{S}(G)$. By (i), we have $|P| = \rho_2(G)$. By (ii) and since $P \subseteq S(G)$, we obtain that $\{N_G[u] : u \in P\}$ is a partition of $V(G)$ into complete sets. Since every 1-packing contains at most one vertex from each complete set, this implies $\rho_1(G) \leq |P|$. Since $\rho_1(G) \geq \rho_2(G)$, it follows $\rho_1(G) = \rho_2(G)$.

In order to prove the necessity, let G satisfy $\rho_1(G) = \rho_2(G)$. Let P be a maximum 2-packing. If some vertex u in P has two non-adjacent neighbors v and w , then $(P \setminus \{u\}) \cup \{v, w\}$ is a 1-packing with more vertices than P , which is a contradiction. Hence all

vertices in P are simplicial. Since no two vertices in P are adjacent, the set P is contained in some transversal Q of $\mathcal{S}(G)$. Since Q is a 1-packing, we obtain $\rho_2(G) = |P| \leq |Q| \leq \rho_1(G) = \rho_2(G)$, that is, $P = Q$, which implies in particular that P is a transversal of $\mathcal{S}(G)$. If $V(G) \setminus \bigcup_{u \in P} N_G[u]$ contains a vertex v , then $P \cup \{v\}$ is 1-packing with more vertices than P , which is a contradiction. Hence $\{N_G[u] : u \in P\}$ is a partition of $V(G)$ into complete sets. Since for every transversal P' of $\mathcal{S}(G)$, the partition $\{N_G[u'] : u' \in P'\}$ equals the partition $\{N_G[u] : u \in P\}$, it follows that every transversal of $\mathcal{S}(G)$ is a maximum 2-packing. Altogether, (i) and (ii) follow. \square

By considering suitable powers of the underlying graph, we obtain the following.

Corollary 2 *A graph G satisfies $\rho_k(G) = \rho_{2k}(G)$ for some positive integer k if and only if*

- (i) *a set of vertices of G is a maximum $2k$ -packing if and only if it is a transversal of $\mathcal{S}(G^k)$, and*
- (ii) *for every transversal P of $\mathcal{S}(G^k)$, the sets $N_{G^k}[u]$ for u in P partition $V(G)$.*

By Corollary 2, it is algorithmically very easy to recognize the graphs G with $\rho_k(G) = \rho_{2k}(G)$.

In view of Theorem 1 and Corollary 2, it makes sense to consider the equality of distance packing numbers $\rho_{k_1}(G)$ and $\rho_{k_2}(G)$ where $k_1 < k_2$ are positive integers that do not satisfy $k_2 = 2k_1$. Our next observation shows that for $k_2 > 2k_1$ such graphs are not very interesting.

Observation 3 *If k_1 and k_2 are positive integers with $k_2 > 2k_1$ and G is a connected graph with $\rho_{k_1}(G) = \rho_{k_2}(G)$, then $\rho_{k_1}(G) = \rho_{k_2}(G) = 1$.*

Proof: Let G be a graph that satisfies $\rho_{k_1}(G) = \rho_{k_2}(G)$. Let P be a maximum k_2 -packing. For a contradiction, we assume that P has more than one element. Let u be a vertex in P . Since P has more than one element, there is a vertex v at distance $k_1 + 1$ from u . Since $k_2 + 1 \geq 2(k_1 + 1)$, every vertex in P has distance more than k_1 from v . Now $P \cup \{v\}$ is a k_1 -packing, which is a contradiction. This completes the proof. \square

Now we consider the case $k_1 < k_2 < 2k_1$ and show that already the smallest possible choice, $k_1 = 2$ and $k_2 = 3$, leads to graphs that will most likely not have a nice structural description.

Theorem 4 *It is NP-hard to determine for a given graph G whether $\rho_2(G) = \rho_3(G)$.*

Proof: We describe a reduction from 3SAT to the considered problem. Therefore, let f be a 3SAT instance with m clauses C_1, \dots, C_m over n boolean variables x_1, \dots, x_n . We construct a graph G whose order is polynomially bounded in terms of n and m such that f is satisfiable if and only if $\rho_2(G) = \rho_3(G)$. For every variable x_i , we create a cycle

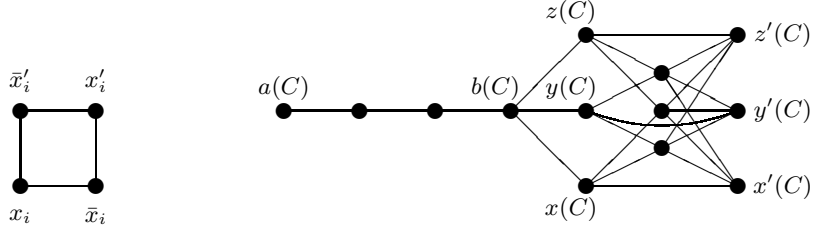


Figure 1: On the left, the cycle $G(x_i) : x_i \bar{x}_i x'_i \bar{x}'_i x_i$ created for the variable x_i . On the right, the graph $G(C)$ created for a clause C with literals x, y , and z , that is, $C = x \vee y \vee z$ and $x, y, z \in \{x_i : i \in [n]\} \cup \{\bar{x}_i : i \in [n]\}$.

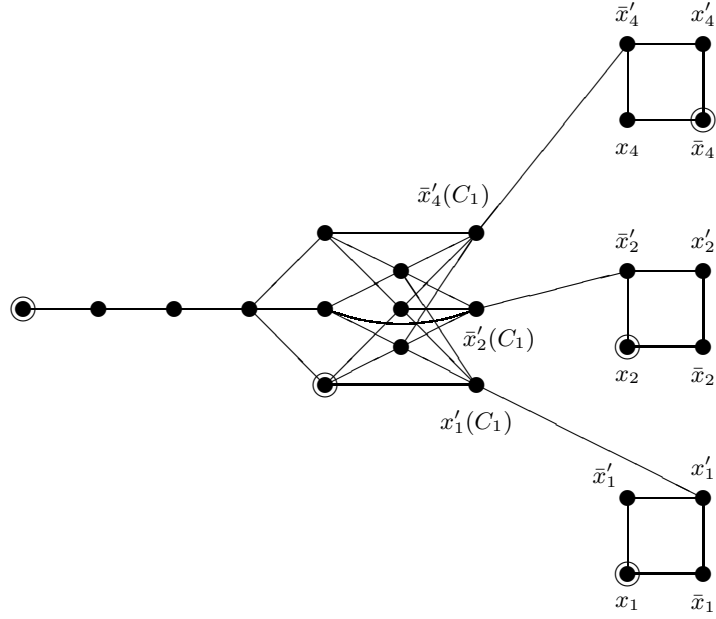


Figure 2: The edges between $G(C_1)$ and $G(x_1) \cup G(x_2) \cup G(x_4)$ created for the clause $C_1 = x_1 \vee \bar{x}_2 \vee \bar{x}_4$. If a satisfying truth assignment sets x_1 and x_2 to true and x_4 to false, the encircled vertices indicate elements of the corresponding 3-packing.

$G(x_i) : x_i \bar{x}_i x'_i \bar{x}'_i x_i$ of length 4 as shown in the left of Figure 1. For every clause C_j , we create a copy $G(C_j)$ of the graph in the right of Figure 1 and denote its vertices as explained in the caption. All graphs $G(x_i)$ and $G(C_j)$ created so far are disjoint. For every clause C with literals x, y , and z , we create the three edges $x'(C)x'$, $y'(C)y'$, and $z'(C)z'$. If, for example, $C_1 = x_1 \vee \bar{x}_2 \vee \bar{x}_4$, then these are the edges $x_1(C)x'_1$, $\bar{x}_2(C)\bar{x}'_2$, and $\bar{x}_4(C)\bar{x}'_4$ as shown in Figure 2. This completes the description of G . It is easy to verify that $\rho_2(G(x_i)) = 1$ and $\rho_2(G(C_j)) = 2$, which implies that $\rho_2(G) \leq n + 2m$. Since $\{a(C_j) : j \in [m]\} \cup \{b(C_j) : j \in [m]\} \cup \{x_i : i \in [n]\}$ is a 2-packing of cardinality $n + 2m$, we obtain $\rho_2(G) = n + 2m$. It remains to prove that f is satisfiable if and only if $\rho_3(G) = n + 2m$.

First, we assume that f is satisfiable and consider a satisfying truth assignment. For every clause C_j , we select a true literal t_j . Note that there may be several choices for

t_j . Now, by construction, the set $\{a(C_j) : j \in [m]\} \cup \{t_j(C) : j \in [m]\} \cup \{x_i : i \in [n] \text{ and } x_i \text{ is true}\} \cup \{\bar{x}_i : i \in [n] \text{ and } x_i \text{ is false}\}$ is a 3-packing of cardinality $n + 2m$, which implies $\rho_3(G) = n + 2m$.

Next, we assume that $\rho_3(G) = n + 2m$. Let P be a maximum 3-packing. Since $\rho_2(G(x_i)) = 1$ and $\rho_2(G(C_j)) = 2$, it follows that P contains exactly one vertex from each $G(x_i)$ and exactly two vertices from each $G(C_j)$. Clearly, we may assume that for every $i \in [n]$, the set P contains exactly one of the two vertices x_i and \bar{x}_i of the cycle $G(x_i)$. Similarly, we may assume that for every $j \in [m]$, the set P contains the vertex $a(C_j)$ and exactly one of the three vertices $x(C_j)$, $y(C_j)$, and $z(C_j)$ where x , y , and z are the three literals in C_j . See Figure 2 for an illustration. We consider the assignment of truth values where the variable x_i is set to true exactly if the vertex x_i belongs to P . If C is a clause and $x \in \{x_i, \bar{x}_i\}$ is a literal in C such that P contains $x(C)$, then $x'(C)$ is adjacent to the vertex x' of $G(x_i)$, and hence P cannot contain the vertex x of $G(x_i)$. More specifically, if $x = x_i$, then P contains the vertex x_i of $G(x_i)$, which means that x_i is set to true, and if $x = \bar{x}_i$, then P contains the vertex \bar{x}_i of $G(x_i)$, which means that x_i is set to false. Altogether, it follows that the truth assignment defined above satisfies f . This completes the proof. \square

A simple modification of the construction in the proof of Theorem 4 allows to establish the following.

Corollary 5 *Let k_1 and k_2 be positive integers with $k_1 < k_2$ and $\lceil (3k_2 + 1)/2 \rceil \leq 2k_1 + 1$. It is NP-hard to determine for a given graph G whether $\rho_{k_1}(G) = \rho_{k_2}(G)$.*

Proof: We apply the following modifications to the graph G constructed in the proof of Theorem 4.

- For every variable x_i , subdivide each of the two edges $x_i \bar{x}'_i$ and $\bar{x}_i x'_i$ exactly $\lceil \frac{k_2}{2} \rceil - 2$ times.
- For each clause C_j with literals x , y , and z ,
 - subdivide the edge incident with $a(C_j)$ exactly $k_2 - 3$ times, and
 - subdivide each of the three edges $b(C_j)x(C_j)$, $b(C_j)y(C_j)$, and $b(C_j)z(C_j)$ exactly $\lfloor \frac{k_2}{2} \rfloor - 1$ times.

Note that after these modifications, the distance between x_i and \bar{x}'_i as well as between \bar{x}_i and x'_i is $\lceil \frac{k_2}{2} \rceil - 1$, the distance between $a(C_j)$ and $b(C_j)$ is k_2 , and the distance between $b(C_j)$ and each of $x(C_j)$, $y(C_j)$, and $z(C_j)$ is $\lfloor \frac{k_2}{2} \rfloor$. Renaming the three neighbors of $b(C_j)$ that do not lie on the path to $a(C_j)$ as $x(C_j)$, $y(C_j)$, and $z(C_j)$, and repeating the very same argument as in the proof of Theorem 4, we obtain that f is satisfiable if and only if the modified graph G' satisfies $\rho_{k_1}(G') = \rho_{k_2}(G')$.

Note that we require $\lceil (3k_2 + 1)/2 \rceil \leq 2k_1 + 1$ instead of just $k_2 < 2k_1$ in order to ensure that $\rho_{k_1}(G(C_j)) = 2$. \square

We proceed to consequences of Theorem 1 for distance matching numbers. Note that a graph G satisfies $\nu_k(G) = \nu_{2k}(G)$ if and only if $\rho_1(L(G)^k) = \rho_2(L(G)^k)$, that is, these graphs can be recognized by a very simple algorithm.

For a positive integer k , a k -unit is a pair (G, e) where G is a connected graph, e is an edge of G , and $\nu_k(G) = 1$. The *boundary* $\partial(G, e)$ of (G, e) is the set of vertices of G that are at distance exactly k from e in G . Note that, since G is connected and $\nu_k(G) = 1$, no vertex of G is at distance more than k from e in G , and the boundary $\partial(G, e)$ is independent.

Corollary 6 *A graph G satisfies $\nu_k(G) = \nu_{2k}(G)$ for some positive integer k if and only if G arises from the disjoint union of k -units $(G_1, e_1), \dots, (G_\ell, e_\ell)$ by arbitrarily identifying vertices in $\bigcup_{i=1}^\ell \partial(G_i, e_i)$, where $\ell = \nu_k(G)$.*

Proof: Let G be a graph.

In order to prove the sufficiency, let G arise in the described way from the k -units (G_i, e_i) . Let P be a maximum 1-packing of $L(G)^k$, that is, P is a set of edges of G that are at pairwise distance more than k in $L(G)$. Since $\nu_k(G_i) = 1$, the set P contains at most one edge from each G_i , which implies that $\nu_k(G) = \rho_1(L(G)^k) = |P| \leq \ell$. By the definition of the boundary, the set $\{e_i : i \in [\ell]\}$ is a 2-packing of $L(G)^k$, and hence $\nu_{2k}(G) = \rho_2(L(G)^k) \geq \ell$, which implies $\nu_k(G) = \nu_{2k}(G)$.

In order to prove the necessity, let G satisfy $\nu_k(G) = \nu_{2k}(G)$, that is, $\rho_1(L(G)^k) = \rho_2(L(G)^k)$. Let P be a maximum 2-packing of $L(G)^k$. By Theorem 1, the set P is a transversal of $\mathcal{S}(L(G)^k)$ and $\{N_{L(G)^k}[e] : e \in P\}$ is a partition of $E(G)$, the vertex set of $L(G)^k$, into sets that are complete in $L(G)^k$. Let $P = \{e_1, \dots, e_\ell\}$ and let $E_i = N_{L(G)^k}[e_i]$ for $i \in [\ell]$. For $i \in [\ell]$, let V_i denote the set of vertices of G that are incident with an edge in E_i , and let $G_i = (V_i, E_i)$. By definition, and since E_i is a complete set in $L(G)^k$, the graph G_i is connected, e_i is an edge of G_i , and $\nu_k(G_i) = 1$, that is, (G_i, e_i) is a k -unit. Note that the graphs G_i are edge-disjoint yet not vertex-disjoint subgraphs of G . If G_i and G_j share a vertex u for some $i \neq j$, and u does not belong to the intersection of the boundaries $\partial(G_i, e_i) \cap \partial(G_j, e_j)$, then the distance in $L(G)^k$ between e_i and e_j is at most 2, which is a contradiction. Hence G arises in the described way from the k -units (G_i, e_i) . This completes the proof. \square

Let G be a graph. A vertex of degree 1 in G is a *leaf* of G . A triangle uvw in G such that the degree of u and v in G is 2 is a *pendant triangle* of G and the edge uv is a *triangle edge* of G .

Corollary 7 (Cameron and Walker [6]) *A connected graph G satisfies $\nu_1(G) = \nu_2(G)$ if and only if G is either a star, or a triangle, or arises from a connected bipartite graph with two non-empty partite sets V_1 and V_2 by*

- attaching at least one and possibly more leaves to each vertex in V_1 , and
- attaching pendant triangles to some vertices in V_2 .

Proof: Let G be a graph that satisfies $\nu_1(G) = \nu_2(G)$. By Corollary 6, the graph G arises from the disjoint union of 1-units by arbitrarily identifying vertices in their boundaries. It follows immediately from the definition that if (G, e) is a 1-unit, then

- either G is a star and $\partial(G, e)$ is the set of leaves of G that are not incident with e ,
- or G is a triangle and $\partial(G, e)$ consists of the vertex that is not incident with e .

The desired structure does not follow immediately. In fact, V_1 is the set of all centers of 1-units that are stars and V_2 is the union of all boundaries (after identification). \square

Our results motivate some questions. In view of Observation 3 it might make sense to consider bounds for $\frac{\rho_{k_1}(G)}{\rho_{k_2}(G)}$ rather than linear relations between $\rho_{k_1}(G)$ and $\rho_{k_2}(G)$. It would be interesting to know whether the decision problems considered in Theorem 4 and Corollary 5 are in NP. We believe that for all positive integers k_1 and k_2 with $k_1 < k_2 < 2k_1$, it is NP-hard to determine for a given graph G whether $\rho_{k_1}(G) = \rho_{k_2}(G)$. Unfortunately, Corollary 5 does not cover all possible cases.

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